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Closed-form solution for Mode I crack acceleration

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Abstract

Two-dimensional semi-infinite crack growth in an otherwise unbounded elastic linear continuum body is considered under the assumption of the idealisation of infinitesimally small scale yielding. The purpose is to model and solve analytically under a closed-form solution by means of a direct method the problem of Mode I crack growth at non-constant velocity for all the range of velocities up to the dilatational wave velocity. As a result, the transonic regime is implicitly taken into account. As many times previously mentioned in the literature, it is then verified that only Mode I sub-Rayleigh crack propagation is physically possible. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Crack acceleration; Fracture, Wiener–Hopf's technique

1. Introduction

Two-dimensional semi-infinite crack growth in an otherwise unbounded elastic linear continuum body is considered under the assumption of the idealisation of infinitesimally small scale yielding. This is a class of related problems to the elastodynamics crack propagation (Broberg, 1999a; Freund, 1990). The opening cracks are usually classified in three modes: the in-plane opening mode crack deformation (Mode I), the in-plane shearing mode (Mode II), and the anti-plane shearing mode (Mode III).

The purpose of this paper is to model and solve analytically the problem of Mode I crack growth at non-constant velocity up to the dilatational wave velocity, including the transonic regime. Mode I is chosen since it is often encountered in engineering structures whereas Mode II and Mode III are most familiar from earthquake slip events.

This class of problem has first been considered by Kostrov (1966). He tackled the problem of non-constant expansion of a finite crack in an infinite body. He assumed Mode III loading and used a method developed within the theory for supersonic flow and assumed that the cohesion modulus (Barenblatt, 1959a,b) is a unique function of time (note that later, Kostrov (1974) and Willis (1989) gave a solution for Mode I under the same assumption). In a series of four papers (Freund, 1972a,b, 1973, 1974), Freund delivered his theory for crack with non-constant velocity. He adopted the central idea of Eshelby's theory

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(Eshelby, 1969) for Mode I crack growth assuming that the solution of a moving crack with non-constant velocity can be found if only the solution of sudden crack propagation at constant velocity is known. Freund made very clever use of superposition techniques. He introduced fundamental solutions to be used as elements in superposition, and he used the principle of negative stress: a crack can be extended by first making a cut with traction on its faces, equalling the stresses present before the cut, and then superpose the same traction, but with opposite sign. Recently, with the help of this superposition scheme (Broberg, 1999b), Broberg gave a solution for intersonic Mode II crack acceleration. His solution is obtained by first considering a self-similar problem, i.e. motion of a semi-infinite crack at constant velocity, under the action of a constant crack free load, appearing behind the crack edge.

Here, unlike Broberg, Eshelby, Freund, Kostrov and Willis, a closed-form solution is directly formulated without any physical asumption a priori on the nature of the singularity around the crack tip. The only anticipated physical assumption is the behaviour of the elastodynamics field far away from the crack tip.

2. Statement of problem

Consider the body of a linear elastic material that contains a half plane elastodynamics crack but that is otherwise unbounded (see Fig. 1). At the initial time, the crack faces are subjected to a suddenly applied pressure and next expand. The material is stress free and at rest everywhere for negative times. The problem is plane strain, symmetrical with respect to the Ox-axis (one will only consider the upper half space $y > 0$) and then the boundary conditions write in the Cartesian Oxy-coordinate axes

$$\begin{aligned} &\sigma_{yy}(x, y = 0^+, t) = \sigma_+(x,t) - \sigma_f(x,t), \\ &\sigma_{xy}(x, y = 0^+, t) = 0, \\ &u_-(x, y = 0^+, t) = u_-(x,t), \end{aligned} \tag{1}$$

where

$$\begin{aligned} &\sigma_+(x,t) = 0 \quad \text{if } x < l(t) \text{ elsewhere } \sigma_+ \text{ is unknown}, \\ &u_-(x,t) = 0 \quad \text{if } x \geqslant l(t) \text{ elsewhere } u_- \text{ is unknown}, \\ &\sigma_f(x,t) = 0 \quad \text{if } x > l(t) \text{ elsewhere } \sigma_f \text{ is known}. \end{aligned} \tag{2}$$

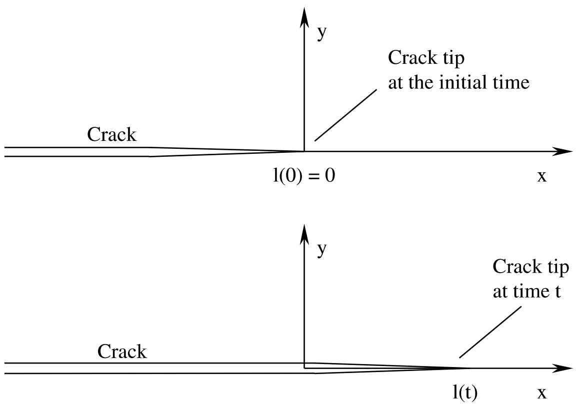

Fig. 1. Geometry of problem.

and where $\boldsymbol{\sigma}$ and $\mathbf{u}$ are the stress tensor and displacement vector respectively which are represented by their Cartesian components, the function $l(t)$ defines the crack length versus time. The applied pressure σ_f is positive, i.e. the material is stretched. The stress ahead σ_+ and the displacement behind u_- the crack tip are two unknown functions. The stress function σ_+ will be authorised to be singular only at the crack tip while the displacement function u_- must be continuous through the crack tip.

At high velocities, the inertial effects need to be taken into account, thus the elastodynamics field must be solution of the following wave equations

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\frac{1}{c_{\mathrm{d}}^2}\frac{\partial^2\phi}{\partial t^2}=0 \quad \text{and} \quad \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{c_{\mathrm{s}}^2}\frac{\partial^2\psi}{\partial t^2}=0, \tag{3}$$

where ϕ and ψ are the dilatational potential and the vector shear potential respectively. The constants c_{d} and c_{s} are the dilatational and shear wave velocities respectively.

Eqs. (1)–(3) together define a mixed boundary value problem which will be solved by means of integral transforms.

3. Application of integral transforms

Solution of the problem proceeds by application of a one-sided Laplace transform over time denoted by an over-bar, and a double-sided Laplace transform over the x coordinate denoted by a star to the governing partial differential equation (3) and boundary conditions (1). The one-sided Laplace transform of a given function $f(t)$ is written as

$$\bar{f}(s)=\int_0^{+\infty} f(t)\exp(-st)\,\mathrm{d}t, \tag{4}$$

where the complex number s has a positive real part, i.e. $\mathrm{Re}(s)\geqslant 0$. The imaginary part of a complex number, say s, will be noted $\mathrm{Im}(s)$.

A particular attention is given here for the double-sided Laplace transform of the stress and displacement based on hindsight. One defines

$$\begin{aligned}\bar{\sigma}^*_{+,f}(p,s)&=s^n\int_{-\infty}^{+\infty}\bar{\sigma}_{+,f}(x,s)\exp(-spx)\,\mathrm{d}x,\\ \bar{u}^*_-(p,s)&=s^{n+1}\int_{-\infty}^{+\infty}\bar{u}_-(x,s)\exp(-spx)\,\mathrm{d}x,\end{aligned} \tag{5}$$

where p is a complex number and the exponent n is an integer. The domain of convergence of these integrals will be subsequently defined.

Then, the transformed solution applied to the boundary conditions on $y=0^+$ gives rise to the following differential system

$$\begin{cases}\left[(1/c_{\mathrm{s}}^2-1/c_{\mathrm{d}}^2)s^2\bar{\phi}^*(p,y,s)+2\frac{\mathrm{d}^2\bar{\phi}^*(p,y,s)}{\mathrm{d}y^2}-2sp\frac{\mathrm{d}\bar{\psi}^*(p,y,s)}{\mathrm{d}y}\right]_{y=0^+}=\frac{1}{\mu s^n}\left(\bar{\sigma}^*_+(p,s)-\bar{\sigma}^*_f(p,s)\right),\\ \left[2sp\frac{\mathrm{d}\bar{\phi}^*(p,y,s)}{\mathrm{d}y}+\frac{\mathrm{d}^2\bar{\psi}^*(p,y,s)}{\mathrm{d}y^2}-s^2p^2\bar{\psi}^*(p,y,s)\right]_{y=0^+}=0,\\ \left[\frac{\mathrm{d}\bar{\phi}^*(p,y,s)}{\mathrm{d}y}-sp\bar{\psi}^*(p,y,s)\right]_{y=0^+}=\frac{\bar{u}^*_-(p,s)}{s^{n+1}},\end{cases} \tag{6}$$

where μ is the Lamé constant.

One seeks for solutions of (3) decaying as $y \to +\infty$ under the form

$$\bar{\phi}^*(p,y,s) = \frac{1}{s^{n+2}} A(p,s)\exp(-s\alpha y), \qquad \bar{\psi}^*(p,y,s) = \frac{1}{s^{n+2}} B(p,s)\exp(-s\beta y), \tag{7}$$

where

$$\alpha = \sqrt{1/c_{\mathrm{d}}^2 - p^2}\ (\mathrm{Re}(\alpha) \geqslant 0) \quad \text{and} \quad \beta = \sqrt{1/c_{\mathrm{s}}^2 - p^2}\ (\mathrm{Re}(\beta) \geqslant 0). \tag{8}$$

The real part of the multivalued functions $\alpha(p)$ and $\beta(p)$ is chosen positive. This defines their associated branch cuts on the real-axis of the complex p-plane, i.e. $]-\infty, -1/c_{\mathrm{d}}] \cup [1/c_{\mathrm{d}}, +\infty[$ and $]-\infty, -1/c_{\mathrm{s}}] \cup [1/c_{\mathrm{s}}, +\infty[$. If p approaches a cut in the right (left) half plane with $\mathrm{Im}(p) \to 0^+$ then the limiting value of either α or β is a negative (positive) value.

With $n = 2$, one obtains the Wiener–Hopf equation of our mixed boundary value problem

$$\bar{\sigma}_+^*(p,s) + \bar{\sigma}_f^*(p,s) = -\mu c_{\mathrm{s}}^2 \frac{R(p)}{\alpha(p)} \bar{u}_-^*(p,s), \tag{9}$$

where

$$R(p) = 4p^2\alpha(p)\beta(p) + (1/c_{\mathrm{s}}^2 - 2p^2)^2 \tag{10}$$

is the Rayleigh function.

Here, we point out that all our reasoning up to Eq. (9) is valid only in a common strip of analyticity in the complex p-plane, if such a strip exists, of $\bar{\sigma}_+^*$ and $\bar{u}_-^*$. A strip of convergence can be found by the knowledge a priori of the asymptotic behaviour of the elastodynamics field in the far field (far away from the crack tip). This follows from the fact that, given a point in the far field, both the displacement and the stress are then governed by the first disturbance which reaches a such point (Freund, 1990; Achenbach, 1993). In the stationary crack problem, $(l(t) = 0,\ \forall t)$, an elementary analysis shows that $\bar{\sigma}_+^*$ and $\bar{u}_-^*$ are regular for $\mathrm{Re}(p) > -1/c_{\mathrm{d}}$ and $\mathrm{Re}(p) < 0$ respectively as illustrated in Fig. 2. The strip of analyticity is then $-1/c_{\mathrm{d}} < \mathrm{Re}(p) < 0$ (Freund, 1990). We claim that, for a moving crack, the strip is unchanged. Indeed, let us suppose the wave fronts in the far field. Given y positive, for large negative values of x nothing is changed with respect to the stationary crack problem. For large positive value of x, even if the crack tip is moving, the first arriving disturbance will be the dilatational cylindrical wave front emanating from the crack tip at the initial time, i.e. the time when the crack begins to expand. This obviously is true if the crack tip velocity is smaller than the dilatational wave velocity. The strip of analyticity of the Wiener–Hopf equation (9) is then defined as follows

$$-1/c_{\mathrm{d}} < \mathrm{Re}(p) < 0. \tag{11}$$

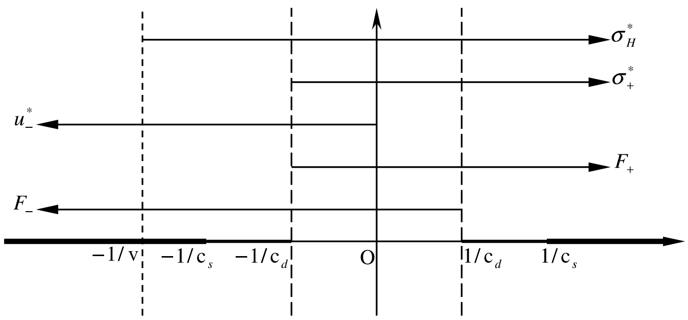

Fig. 2. The complex p-plane showing the location of the branch cuts (in bold) of the analytic functions involved in the Wiener–Hopf equation (9), i.e. $F_\pm$, $\bar{\sigma}_+^*$, $\bar{u}_-^*$ and $\bar{\sigma}_H^*$, which applies in the strip $-1/c_{\mathrm{d}} < \mathrm{Re}(p) < 0$.

In a first time, one will try to seek for the Green's function, i.e. the response to an applied spatially uniform pressure written under the form

$$\sigma_H(x,t) = \sigma_0 H(t) H(l(t) - x), \tag{12}$$

where H refers to the Heaviside step function. More general functions would not be suitable for direct analytic treatment.

4. Case of an applied spatially uniform pressure

The boundary conditions (1) write as follows by using Eq. (12)

$$\begin{aligned} &\sigma_{yy}(x,0,t) = \sigma_+(x,t) - \sigma_0 H(t) H(l(t)-x), \\ &\sigma_{xy}(x,0,t) = 0, \\ &u_-(x,0,t) = u_-(x,t). \end{aligned} \tag{13}$$

Since the function $l(t)$, which determines the crack length, is necessarily continuous, positive and monotonic, it has then an inverse function, say η, with the following identities

$$t = \eta(x), \quad l(\eta(x)) = x, \quad \eta(l(t)) = t \text{ and } \eta'(x) = 1/l'(t) \text{ with } \eta(x) = 0 \text{ if } x \leqslant 0. \tag{14}$$

The prime denotes the derivative, which can be discontinuous. Thus the derivative $l'(t)$ can admit jumps. Note that one will have the condition

$$l'(t) < c_\mathrm{d} \quad \forall t, \tag{15}$$

which traduces that a supersonic crack is not considered. This does not exclude, however, a transonic crack, i.e. $c_\mathrm{s} < l'(t) < c_\mathrm{d}$.

Now, with the choice of $n = 1$ in Eqs. (5) and (7), the Wiener–Hopf equation can be written as follows

$$\begin{aligned} &\int_{-\infty}^{+\infty} \bar{\sigma}_+(x,s) \exp -s(px)\,\mathrm{d}x + \frac{\sigma_0}{p}\int_{-\infty}^{+\infty} \eta'(x) \exp -s[\eta(x) + px] H(x)\,\mathrm{d}x \\ &\quad = -\mu c_\mathrm{s}^2 \frac{R(p)}{\alpha(p)} \int_{-\infty}^{+\infty} \bar{u}_-(x,s) \exp -s[\eta(x) + px]\,\mathrm{d}x \end{aligned} \tag{16}$$

with

$$\begin{aligned} &\bar{\sigma}^*_H(p,s) = \frac{1}{s}\frac{\sigma_0}{p}\int_{-\infty}^{+\infty} \eta'(x) \exp -s[\eta(x) + px] H(x)\,\mathrm{d}x, \\ &\bar{\sigma}^*_+(p,s) = s\int_{-\infty}^{+\infty} \bar{\sigma}_+(x,s) \exp -s(px)\,\mathrm{d}x, \\ &\bar{u}^*_-(p,s) = s^2\int_{\infty}^{+\infty} \bar{u}_-(x,s) \exp -s(px)\,\mathrm{d}x, \end{aligned} \tag{17}$$

where the method to obtain $\bar{\sigma}^*_H$ is detailed in Appendix A.

The Wiener–Hopf technique (Nobel, 1958) consists to factorise (see Appendix B) the α and R functions on the right-hand side of Eq. (16) in order to obtain an appropriate form whose one side is a regular function in $\mathrm{Re}(p) > -1/c_\mathrm{d}$, say + domain, and the other is a regular function in $\mathrm{Re}(p) < 0$, say − domain. These manipulations transform Eq. (16) under the form

$$\overline{\Omega}^*_+(p,s) + \overline{\Omega}^*_f(p,s) = -\lambda \frac{\overline{U}^*_-(p,s)}{F_+(p)F_-(p)}, \tag{18}$$

where $\lambda = 2\mu(1 - (c_s/c_d))$, with

$$\overline{\Omega}^*_+(p,s) = \frac{1}{s}\bar{\sigma}^*_+(p,s), \overline{\Omega}^*_H(p,s) = s\bar{\sigma}^*_H(p,s) \quad \text{and} \quad \overline{U}^*_-(p,s) = \frac{1}{s^2}\bar{u}^*_-(p,s). \tag{19}$$

One can rewrite Eq. (18) as follows

$$F_+(p)\left[\overline{\Omega}^*_+(p,s) + \overline{\Omega}^*_f(p,s)\right] = -\lambda\frac{\overline{U}^*_-(p,s)}{F_-(p)}. \tag{20}$$

At this stage, the resolution of Eq. (20) cannot be pursued further without changes on the nature of the applied spatially uniform pressure defined in Eq. (13) (see Appendix C). Thus to avoid all difficulties in our proof, one replaces the constant applied stress σ_0 with a time dependant stress $f(t)$ which is positive, bounded, i.e. $\exists\sigma_0/f(t) < \sigma_0\ \forall t$, and such that its support is bounded, i.e. $\exists t_0/f(t) = 0\ \forall t \geqslant t_0$. In this case, one is no longer seeking for the Green's function of the problem associated to the Heaviside step function $H(t)$, but the nature of the applied pressure becomes more realistic since it is then bounded over time. One defines then $\overline{\Omega}^*_f$, which replaces $\overline{\Omega}^*_H$, as follows

$$\overline{\Omega}^*_f(p,s) = \frac{1}{p}\int_{-\infty}^{+\infty} f(\eta(x))\eta'(x)\exp -s[\eta(x) + px]H(x)\,\mathrm{d}x. \tag{21}$$

The only one singularity of the mixed function $\bar{\Omega}^*_f$ to take into account in the + domain is then the simple pole at $p = 0$ (see Appendix C), which is analytic elsewhere. This pole can, however, be removed by writing

$$F_+(p)/p = [F_+(p) - F_+(0)]/p + F_+(0)/p, \tag{22}$$

where the first (second) term on the right-hand side of Eq. (22) is analytic in the $+(-)$ domain.

Now, Eq. (20) rearranges in the following manner

$$\overline{\Omega}^*_f(p,s)[F_+(p) - F_+(0)] + F_+(p)\overline{\Omega}^*_+(p,s) = -\overline{\Omega}^*_f(p,s)F_+(0) - \lambda\frac{\bar{U}^*_-(p,s)}{F_-(p)}. \tag{23}$$

The left (right)-hand side of Eq. (23) is regular in the $+(-)$ domain. Because of the equality in the strip of overlap both sides of Eq. (23) represents one and the same entire function according to the principle of analytical continuation (Mclachlan, 1953).

At this stage, one recalls that in the particular case of a stationary crack with $f(t) = \sigma_0 H(t)$, $\bar{\Omega}^*_f(\equiv \overline{\Omega}^*_H)$ reduces to σ_0/p and

$$\lim_{|p|\to\infty}\left|\overline{\Omega}^*_+(p,s)\right| \propto |p|^{-1/2}, \quad \lim_{|p|\to\infty}\left|\bar{U}^*_-(p,s)\right| \propto |p|^{-3/2} \quad \text{with} \quad \lim_{|p|\to\infty}|F_\pm(p)| \propto |p|^{-1/2}, \tag{24}$$

where $|p|$ is the modulus of p. Since it can be verified that both sides of Eq. (23) tend at zero as $|p| \to \infty$ then, from Liouville's theorem, one concludes that both sides are identically null. No we claim that this result is unchanged if $\overline{\Omega}^*_f$ has the form

$$\frac{1}{p}\int_0^{+\infty} f(\eta(x))\eta'(x)\exp -s[\eta(x) + px]\,\mathrm{d}x, \tag{25}$$

since the convergence is stronger than $1/|p|$ as $|p| \to \infty$ (see Appendix D).

Then the solution of Eq. (23) writes

$$\begin{aligned} \overline{\Omega}^*_+(p,s) &= \overline{\Omega}^*_f(p,s)[F_+(0)/F_+(p) - 1], \\ \bar{U}^*_-(p,s) &= -\frac{1}{\lambda}F_+(0)F_-(p)\overline{\Omega}^*_f(p,s). \end{aligned} \tag{26}$$

Focusing our attention in a first time on the stress, one can rewrite the first relation of Eq. (26) under the form

$$\int_{-\infty}^{+\infty} s\bar{\sigma}_+(\omega,s)\exp -s(p\omega)\,\mathrm{d}\omega = \int_0^{+\infty} f(\eta(v))\eta'(v)\Sigma_+(p)\exp -s[\eta(v)+pv]\,\mathrm{d}v, \tag{27}$$

with

$$\Sigma_+(p) = \frac{1}{p}[F_+(0)/F_+(p) - 1], \tag{28}$$

where ω and v are free variables which just describe the domain of integration, and the constant $F_+(0) = (1-\kappa)^{-1}\sqrt{c_\mathrm{d}(1-2\kappa)/2}$ where κ represents Poisson's ratio.

4.1. Inversion of the transforms: extraction of σ_+

Let us take the double-sided inverse Laplace transform of Eq. (27) both on the left- and right-hand sides

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{d}\omega\frac{1}{2\mathrm{i}\pi}\int_{\varepsilon-\mathrm{i}\infty}^{\varepsilon+\mathrm{i}\infty} s\bar{\sigma}_+(\omega,s)\exp -s[p(\omega-x)]\,\mathrm{d}p\\ &\quad=\int_0^{+\infty} f(\eta(v))\eta'(v)\,\mathrm{d}v\frac{1}{2\mathrm{i}\pi}\int_{\varepsilon-\mathrm{i}\infty}^{\varepsilon+\mathrm{i}\infty}\Sigma_+(p)\exp -s[\eta(v)+p(v-x)]\,\mathrm{d}p\end{aligned} \tag{29}$$

where 2π refers to the length of the unit circle and ε is a real number in the strip defined by Eq. (11). Both the inner integrals on the left- and right-hand sides of Eq. (29) reduce from Cagniard's technique (Aki and Richards, 1980) with the change of variables $t = p(\omega - x)$ and $t = \eta(v) + p(v - x)$ respectively (see Appendix E). One obtains

$$\frac{1}{\mathrm{i}\pi}\int_{-\infty}^{+\infty}\frac{1}{\omega-x}\bar{\sigma}_+(\omega,s)\,\mathrm{d}\omega = \frac{1}{\pi}\int_0^{+\infty}\frac{f(\eta(v))\eta'(v)}{v-x}\mathrm{Im}[\Sigma_+(p)]H(t-\eta(v))\,\mathrm{d}v, \tag{30}$$

which we rewrite for brevity as follows

$$\int_{-\infty}^{+\infty}\mathrm{left}(\omega)\,\mathrm{d}\omega = \int_0^{+\infty}\mathrm{right}(v)\,\mathrm{d}v. \tag{31}$$

But $H(t-\eta(v)) = 0$ if $t < \eta(v)$ or $v > l(t)$. Consequently,

$$\mathrm{right}(v) = 0 \quad \text{for } v > l(t), \tag{32}$$

and

$$\int_{l(t)}^{+\infty}\mathrm{right}(v)\,\mathrm{d}v = 0. \tag{33}$$

One has then the following result

$$\int_{-\infty}^{+\infty}\mathrm{left}(\omega)\,\mathrm{d}\omega = \int_0^{l(t)}\mathrm{right}(v)\,\mathrm{d}v. \tag{34}$$

Now, let us consider the integral on the left-hand side of Eq. (30) with ω as a complex variable. To evaluate it, one chooses a contour of integration as shown in Fig. 3. From Cauchy's theorem the integral along a closed contour is null if no poles are encircled. This is traduced by

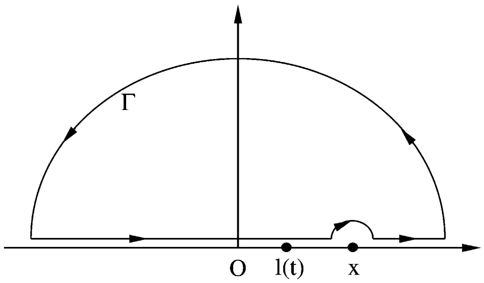

Fig. 3. Contour of integration in the complex ω-plane to reduce the left-hand side of Eq. (30). The rays of the large and small half-circles tend at infinity and zero respectively.

$$\int_{\Gamma} \cdots \, \mathrm{d}\omega + \int_{-\infty}^{l(t)} \cdots \, \mathrm{d}\omega + \int_{l(t)}^{+\infty} \cdots \, \mathrm{d}\omega - \mathrm{i}\pi \mathrm{Res}[\mathrm{left}(\omega)]_{\omega=x} = 0, \tag{35}$$

where it is anticipated that the real pole at $\omega = x$ is the only one singularity in the integrand and σ_+ is expected to be analytic in the complex ω-plane.

Now, from Jordan's lemma

$$\int_{\Gamma} \ldots \, \mathrm{d}\omega = 0, \tag{36}$$

in accordance with the principle of causality, i.e. given t, $\sigma_+(|\omega|, t) = 0$, $\forall |\omega| > c_\mathrm{d} t$.

The second term of Eq. (35) is null by assumption since

$$\sigma_+(\omega, t) = 0 \quad \text{for } \omega < l(t). \tag{37}$$

The third term is equal to the right-hand side of Eq. (34), and then

$$-\mathrm{i}\pi \mathrm{Res}\left[\frac{1}{\omega - x} \sigma_+(\omega, t)\right] = -\mathrm{i}\pi \sigma_+(x, t). \tag{38}$$

Note from Eq. (37) that the Residue is null in Eq. (38) if $x < l(t)$. Finally, one has

$$\begin{aligned} \sigma_+(x, t) &= \frac{1}{\pi} \int_0^{l(t)} \frac{f(\eta(v))\eta'(v)}{v - x} \mathrm{Im}\left[\Sigma_+\left(\frac{t - \eta(v)}{v - x}\right)\right] \mathrm{d}v \quad && \text{for } x > l(t), \\ &= 0 && \text{elsewhere.} \end{aligned} \tag{39}$$

One can simplify Eq. (39) with the help of Eq. (14) by writing

$$\tau = \eta(v), \quad \omega = l(\tau) \text{ and } \tau = \eta(l(\tau)) \tag{40}$$

and deduces

$$\sigma_+(x, t) = \frac{1}{\pi} \int_0^t \frac{f(\tau)}{t - \tau} \mathrm{Im}[\Pi_+(p)] \, \mathrm{d}\tau \, H(x - l(t)), \tag{41}$$

where

$$\Pi_+(p) = p\Sigma_+(p) \quad \text{and} \quad p = \frac{t - \tau}{l(\tau) - x}. \tag{42}$$

The function Π_+ becomes imaginary if the square root function $\alpha_+(p) = \sqrt{1/c_\mathrm{d} + p}$ becomes imaginary. This is realised if $p < -1/c_\mathrm{d}$, i.e. from $\tau = 0$ to the only one time τ_0 which is solution of

$$V(\tau_0) = \frac{x - l(\tau_0)}{t - \tau_0} = c_\mathrm{d} \quad \text{for } \tau_0 \in [0, t]. \tag{43}$$

There is only one τ_0 since the function $l(t)$ is necessarily increasing and $l'(\tau) < c_\mathrm{d}$, $\forall \tau \in [0, t]$. Consequently, one has not to integrate up to the time t but up to τ_0. Such a value only exists if

$$x > l(t) \quad \text{and} \quad x \leqslant c_\mathrm{d} t \tag{44}$$

with the important particular case on the crack tip

$$\tau_0 \to t \quad \text{if } x \to l(t). \tag{45}$$

The support of the domain of integration in Eq. (41) can be then clarified and the integral written as follows

$$\sigma_+(x, t) = \int_0^{\tau_0} \ldots \,\mathrm{d}\tau H(t - \tau_0) H(x - l(t)) H(c_\mathrm{d} t - x) \tag{46}$$

with

$$\tau_0 \in [0, t] / V(\tau_0) = c_\mathrm{d}, \tag{47}$$

as defined in Eq. (43), and where

$$l'(\tau) < c_\mathrm{d}. \tag{48}$$

We point out that in transonic regime

$$\exists \tau_0 \in [0, t] / -1/c_\mathrm{s} < p(= -1/V(\tau_0)) < -1/c_\mathrm{d} \tag{49}$$

and the integrand of S_+ in Eq. (B.4) becomes singular. But it can be observed that S_+ and S_- are, however, never singular simultaneously. One can calculate this exceptional case by employing

$$S_+(p) = \frac{S(p)}{S_-(p)}, \tag{50}$$

where one chooses in the denominator the S-function, here S_-, which is not defined by a singular integral.The transonic regime is implicitly taken into account without anymore modification. This would not have been the case if one had started with a moving crack at constant velocity, since a pole appears then between $-1/c_\mathrm{s}$ and $-1/c_\mathrm{d}$. Thus its contribution needs to be evaluated separately to extract the Cauchy principal value of S_+ and the Wiener-Hopf equation must be rearranged in consequence (Broberg, 1999a,b).

At this stage, the result for the stationary case, i.e. $l(t) = 0\ \forall t$, with $f(t) = \sigma_0$, can be refined with the change of variable

$$\theta = t - \tau, \tag{51}$$

and then the integral rewrites in Eq. (46) as follows

$$\sigma_+(x, t) = \frac{-1}{\pi x} \int_{x/c_\mathrm{d}}^{t} \mathrm{Im}\left[\Sigma_+\left(-\frac{\theta}{x}\right)\right] \mathrm{d}\theta H(c_\mathrm{d} t - x). \tag{52}$$

Expression (52) is exactly the response to

$$\sigma_0 H(t) H(-x) \tag{53}$$

which was first given by Maue (1954).

Now, let us study more in detail what is happening at the crack tip which is the only singular point of the problem? Indeed, the calculations show the evidence of a singularity at $x = l(t)$ since

$$\lim_{\tau \to t} \left[\frac{t - \tau}{l(\tau) - x}\right]_{x = l(t)} = -\frac{1}{l'(t)} \tag{54}$$

leads to the following result

$$
\begin{aligned}
f(t)\lim_{\tau\to t}\left\{\frac{1}{t-\tau}\mathrm{Im}\left[\Pi_{+}\left(p=\frac{t-\tau}{l(\tau)-x}\right)\right]\right\}_{x=l(t)} &= f(t)\mathrm{Im}\left[\Pi_{+}\left(p=-\frac{1}{l'(t)}\right)\right]\lim_{\tau\to t}\left\{\frac{1}{t-\tau}\right\},\\
&=\infty \quad \text{if } t<t_0,\\
&=0 \quad \text{if } \quad t\geqslant t_0,
\end{aligned}
\tag{55}
$$

where one recalls that t_0 is the time when the applied pressure vanishes.

One knows from Eq. (45) that if $x \to l(t)$ then the domain of integration is $[0,t]$. Thus the stress is singular at the crack tip while $t < t_0$ since a logarithmic singularity is observed in Eq. (55). If the crack stops before then the stress is singular up to t_0, else if it stops after t_0, say t_1, then the stress becomes finite from t_0 and decreases up to t_1. This is not in contradiction with Maue's results (1954). A particular important study leading to the dynamic stress intensity factor follows but this was not our aim. One can recall, however, that a crack can both open and propagate only if both the stress σ_+ just in front of the crack tip and the displacement u_- just behind work together. At the starting point, i.e. $t = 0$ with $l(t) = 0$, this condition can be verified from the stationary case. The tensile stress is then positive. Thus, when the crack is further propagating, if one of them, i.e. σ_+ or u_-, change sign then crack propagation is not possible. It is expedient to see from Eq. (B.2) that $\mathrm{Im}[\Pi_+(p = -1/l'(t))]$ change sign when $l'(t) = c_R$. Thus, if $l'(t)$ is continuous and monotonic increasing, no solution is possible after the time when the Rayleigh wave velocity is reached. Suppose now that $l'(t)$ increases abruptly from a sub-Rayleigh to an intersonic velocity. Then, from Eqs. (8), (10), (50) and (B.2), it is straightforward to observe a change of sign. One verifies the well-known result that only Mode I sub-Rayleigh crack propagation is physically possible. Note that in parallel works one has written the problem for Mode II and confirmed that intersonic regime is possible.

Of course, one has assumed the cinematic of the crack tip by a given hypothetical equation of motion $l(t)$ but obviously, only the establishment of a crack tip equation of motion in a theory of crack dynamics can answer definitively about either the advance or the stop of cracks.

4.2. Inversion of the transforms: extraction of u_-

The described previous procedure for the calculation of σ_+ can be identically repeated to seek for u_-. Note, however, that due to the definitions in Eq. (5), one has directly the displacement velocity $u'_- \equiv \partial u(x,t)/\partial t$ and not u_-.

The difference is that here the square root function $\alpha_-(p) = \sqrt{1/c_d - p}$ becomes imaginary if $p > 1/c_d$. Starting back from Eq. (26), one gives the following final result

$$
u'_-(x,t) = \frac{1}{\pi}\int_{\eta(x)}^{t_0} \frac{f(\tau)}{t-\tau}\mathrm{Im}[\Delta_-(p)]\,\mathrm{d}\tau H(l(\tau_0)-x)H(t-\tau_0)H(c_d t + x),
\tag{56}
$$

where

$$
\Delta_-(p) = \frac{-1}{\lambda}F_+(0)F_-(p), \quad p = \frac{t-\tau}{l(\tau)-x}
\tag{57}
$$

with

$$
\tau_0 \in [0,t]/V(\tau_0) = -c_d.
\tag{58}
$$

Note that, in transonic regime, Eq. (50) must be used with a change of sign, i.e. $S_-(p) = S(p)/S_+(p)$.

In general, given $x < l(t)$, if $\tau \to \eta(x)$ then the integrand in Eq. (56) vanishes in virtue of Eq. (24). However, on the crack tip, $\tau_0 \to t$, and this integrand becomes singular. But in virtue of Eq. (45), the domain of integration vanishes while this was not the case for the stress σ_+ in Eq. (46). The displacement velocity u'_- is then null as soon as the displacement since u_- is null at the initial time. This is perfectly in accordance with the continuity of the displacement through a crack tip since u_- is by assumption null in front of the crack. One can then verify

$$\lim_{x\to l(t)} u_-(x,t) = u_-(l(t),t) = 0. \tag{59}$$

It does not matter that the integrand in Eq. (56) is singular or not, the boundary condition (59) remains valid. More generally, Eqs. (55) and (59) together are consistent with the physical boundary conditions in the vicinity of a crack tip.

5. Conclusion

The elastodynamics field ahead a two-dimensional semi-infinite crack in-plane opening mode with a non-uniform velocity in an unbounded medium has been formulated under a closed-form by means of a direct method. As a result, the developed method implicitly takes into account the transonic regime.

However, although only the Mode I was considered, the developed method is also suitable for Modes II and III propagation.

Acknowledgements

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Appendix A. Note on the result: $\delta(x - l(t)) = \eta'(x)\delta(t - \eta(x))$

Let us start with the Dirac delta function

$$\delta(x - l(t)). \tag{A.1}$$

The double-sided Laplace transform over the x coordinate of Eq. (A.1) writes

$$\exp -s[pl(t)]. \tag{A.2}$$

The one-sided Laplace transform over time of Eq. (A.2) writes

$$\int_0^{+\infty} \exp -s[pl(t) + t]\,\mathrm{d}t. \tag{A.3}$$

If in Eq. (A.3) one operates the change of variable

$$t = \eta(x) \quad \text{with } x = l(t), \tag{A.4}$$

then

$$\mathrm{d}t = \eta'(x)\,\mathrm{d}x, \tag{A.5}$$

and Eq. (A.3) rewrites

$$\begin{aligned}\int_{-\infty}^{+\infty} \eta'(x) \exp -s[px + \eta(x)]H[\eta(x)]\,\mathrm{d}x &= \int_{-\infty}^{+\infty} \eta'(x) \exp -s[px + \eta(x)]H(x)\,\mathrm{d}x,\\ &= \int_0^{+\infty} \eta'(x) \exp -s[px + \eta(x)]\,\mathrm{d}x.\end{aligned} \tag{A.6}$$

This is exactly the transform of

$$\eta'(x)\delta(t - \eta(x)). \tag{A.7}$$

One has then the following relation between Eq. (A.1) and (A.7)

$$\delta(x - l(t)) = \eta'(x)\delta(t - \eta(x)). \tag{A.8}$$

One can repeat the same reasoning to write the Laplace transform of $H(l(t) - x)$ under the form

$$\int_0^{+\infty} \frac{\eta'(x)}{sp} \exp -s[px + \eta(x)]\,\mathrm{d}x, \tag{A.9}$$

where only the product sp differs from Eq. (A.6).

Appendix B. Factorisation of the Wiener–Hopf equation

The factorisation of the function α is elementary. One has

$$\alpha_\pm = \sqrt{1/c_\mathrm{d} \pm p}, \qquad \alpha = \alpha_+\alpha_-. \tag{B.1}$$

The factorisation of the Rayleigh function R is more complicated but is now became standard (Freund, 1990; Achenbach, 1993). One only gives the following final result

$$F_\pm = \frac{\alpha_\pm}{(1/c_\mathrm{R} \pm p)S_\pm(p)} \tag{B.2}$$

with

$$S = \frac{R(p)}{\kappa(1/c_\mathrm{R}^2 - p^2)} = S_+S_-, \qquad \kappa = 2(1/c_\mathrm{s}^2 - 1/c_\mathrm{d}^2), \tag{B.3}$$

where

$$S_\pm = \exp\left\{\frac{-1}{\pi}\int_{1/c_\mathrm{d}}^{1/c_\mathrm{s}} tg^{-1}\left[\frac{4z^2\sqrt{(z^2 - 1/c_\mathrm{d}^2)(1/c_\mathrm{s}^2 - z^2)}}{(1/c_\mathrm{s}^2 - 2z^2)}\right]\frac{\mathrm{d}z}{z \pm p}\right\}, \tag{B.4}$$

and the Rayleigh wave velocity c_R is solution of the Rayleigh equation (10).

Appendix C. Note on the analyticity of $\overline{\Omega}_f^*$

Let us write $\bar{\Omega}_f^*$ from Eqs. (19) and (21) as follows

$$\overline{\Omega}_f^*(p,s) = \frac{1}{p}\int_0^{+\infty} f(\eta(x))\eta'(x)\exp -s[\eta(x) + px]\,\mathrm{d}x \tag{C.1}$$

with

$$\overline{\Omega}_H^*(p,s) = \frac{\sigma_0}{p}\int_0^{+\infty} \eta'(x)\exp -s[\eta(x) + px]\,\mathrm{d}x. \tag{C.2}$$

From Appendix A, the integrals above can write under the form

$$\overline{\Omega}_f^*(p,s) = \frac{1}{p}\int_0^{+\infty} f(t)\exp -s[pl(t) + t]\,\mathrm{d}t \tag{C.3}$$

and

$$\overline{\Omega}_H^*(p,s) = \frac{\sigma_0}{p}\int_0^{+\infty} \exp -s[pl(t) + t]\,\mathrm{d}t \tag{C.4}$$

respectively.

The pole at $p = 0$ is obvious but, is $\overline{\Omega}_f^*$ analytic elsewhere?

In a first time, consider the function f as a constant, say σ_0, as one chose at the beginning in Eq. (13) and carried out in Eq. (C.2).

Two cases can then be considered: the crack tip either stops after a transient period or accelerates up to a quasi-steady-state regime and never stops. The first case traduces by

$$0 \leqslant l(t) \leqslant l(t_0) \tag{C.5}$$

and the second by

$$0 \leqslant l'(t) < c_\mathrm{d}, \quad \text{with } \lim_{t\to\infty} l'(t) = v(< c_\mathrm{d}), \text{ i.e. } \forall\varepsilon > 0,\ \exists t_0/t > t_0 \Rightarrow \left|l'(t) - v\right| < \varepsilon, \tag{C.6}$$

where t_0 and v are constants, and the derivative $l'(t)$ is monotonic.

In the first case, given s positive, $\forall p$ the integrand in Eq. (C.4) is summable (Arsac, 1966) and then the integral is finite. One concludes that if the crack stops then the integral in Eq. (C.2) has no poles in the complex p-plane and $\overline{\Omega}_H^*$ is then regular everywhere.

The second case belongs to the class of self-similar problems when the steady-state regime is taken into account only. Although it will not be considered here, one can just say that the integration in Eq. (C.4) becomes elementary. Indeed, with $l(t) = vt$, $\forall t$, Eq. (C.4) becomes

$$\overline{\Omega}_H^*(p,s) = \frac{\sigma_0}{sp(vp+1)}. \tag{C.7}$$

It occurs a pole at $p = -1/v$ and the rearrangement of the Wiener-Hopf equation is no longer so easy (Baker, 1962). Baker partially solved the problem by rewriting it with respect to the crack tip coordiantes. Partially, because he could not take into account the transonic regime.

As a more realistic example, one could take $l(t) = v(\sqrt{t^2 + t_0^2} - t_0)$ which linearly behaves as $v(t - t_0)$ if $t \to \infty$. It is then immediate to observe that the integrand in Eq. (C.4) is not summable if $\mathrm{Re}(p) \leqslant 1/v$, and $\overline{\Omega}_H^*$ becomes then regular in $\mathrm{Re}(p) > -1/v$ only (see Fig. 2). The left- and right-hand sides of the modified Wiener-Hopf equation (23) are now regular in $\mathrm{Re}(p) > -1/v$ and $-1/v < \mathrm{Re}(p) < 0$ respectively (with $v < c_\mathrm{d}$). Consequently, Eq. (11) is unmodified but Liouville's theorem, which will help us in the next, loses all its utility.

This motivated us to replace $\sigma_0 H(t)$ with a function $f(t)$, which has a bounded support, and define $\overline{\Omega}_f^*$ in Eq. (21) as carried out in Eq. (C.1). The integrand in Eq. (C.3) becomes then summable and the function $\overline{\Omega}_f^*$ is analytic in the whole complex p-plane, except of course at $p = 0$.

Appendix D. Note on the behaviour of $\overline{\Omega}_f^*$ at infinity

From Appendix A, the function $\overline{\Omega}_f^*(p,s)$ can write under the form

$$\frac{1}{p}\int_0^{+\infty} f(t)\exp -s[pl(t) + t]\,\mathrm{d}t. \tag{D.1}$$

But, let us consider first the following integral

$$\frac{1}{p}\int_0^{+\infty} \exp -s[pl(t) + t]\,\mathrm{d}t. \tag{D.2}$$

Since by assumption $l'(t) < c_\mathrm{d}$, then $\forall t$, $l(t) < c_\mathrm{d}t$ and, given s positive, it follows the inequalities

$$\frac{1}{|p|}\int_0^{+\infty} \exp -st[pc_\mathrm{d} + 1]\,\mathrm{d}t < \frac{1}{|p|}\int_0^{+\infty} \exp -s[pl(t) + t]\,\mathrm{d}t \leqslant \frac{1}{|p|}\int_0^{+\infty} \exp(-st)\,\mathrm{d}t, \tag{D.3}$$

or

$$\frac{1}{s|p|(|p|c_{\mathrm{d}}+1)} < \frac{1}{|p|}\int_0^{+\infty} \exp -s[|p|l(t)+t]\,\mathrm{d}t \leqslant \frac{1}{s|p|} \tag{D.4}$$

from which, one immediately concludes that the integral in Eq. (D.2) tends at zero if one lets $|p|$ tend to infinity. Given the function f positive and bounded, say $f(t) < \sigma_0$, $\forall t$, then the integral in Eq. (D.1) also tends at zero.

Appendix E. Inversion of the left (1) and right (2) hand sides of Eq. (29)

(1) Inversion of $\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} s\bar{\sigma}_+(\omega,s)\exp -s[p(\omega-x)]\,\mathrm{d}p$

With the change of variable first introduced by Cagniard in seismology (Cagniard, 1939)

$$p(\omega - x) = t \tag{E.1}$$

and the contour of integration shown in Fig. 4, one deduces from Cauchy's theorem

$$\begin{aligned} s\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \bar{\sigma}_+(\omega,s)\exp -s[p(\omega-x)]\,\mathrm{d}p &= 2s\int_0^{+\mathrm{i}\infty} \bar{\sigma}_+(\omega,s)\frac{\mathrm{d}p}{\mathrm{d}t}\exp(-st)\,\mathrm{d}t, \\ &= \frac{2s}{\omega-x}\int_0^{+\infty} \bar{\sigma}_+(\omega,s)\exp(-st)\,\mathrm{d}t. \end{aligned} \tag{E.2}$$

The integral on the right-hand side of Eq. (E.2) is trivial. One obtains

$$\frac{2}{\omega-x}\bar{\sigma}_+(\omega,s) \tag{E.3}$$

which is nothing else but the one-sided Laplace transform over time of $(2/(\omega-x))\sigma_+(\omega,t)$. We stress here that it is anticipated that σ_+ can be singular only on $\omega = x$.

(2) Inversion of $\frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \Sigma_+(p)\exp -s[\eta(v)+p(v-x)]\,\mathrm{d}p$

How to invert this kind of integral is now became standard in seismology (Aki and Richards, 1980) and elastodynamics (Achenbach, 1993). Again, as above, from the new change of variable

$$\eta(v) + p(v-x) = t \tag{E.4}$$

and the contour of integration shown in Fig. 5, one deduces from Cauchy's theorem

$$\frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \Sigma_+(p)\exp -s[\eta(v)+p(v-x)]\,\mathrm{d}p = \int_0^{+\infty}\frac{1}{\pi}\frac{1}{v-x}\mathrm{Im}[\Sigma_+(p(t))]H(t-\eta(v))\exp(-st)\,\mathrm{d}t, \tag{E.5}$$

where

$$p(t) = \frac{t-\eta(v)}{v-x} \tag{E.6}$$

and Scharwz's reflexion principle, i.e. $\overline{\Sigma_+(p)} = \Sigma_+(\bar{p})$ has been used. The over-bar refers here to the conjugate imaginary part.

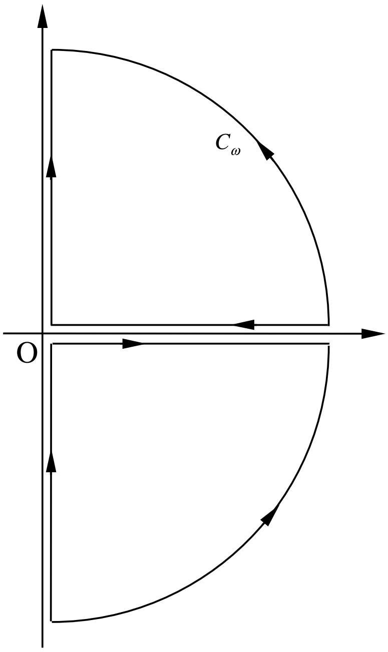

Fig. 4. Contour of integration in the complex p-plane to reduce the inner integral on the right-hand side of Eq. (29). In bold are represented the branch cuts of α_+ and β_+, and their branch points at $-1/c_d$ and $-1/c_s$ respectively. The rays of the truncated large half-circle and small circle tend at infinity and zero respectively.

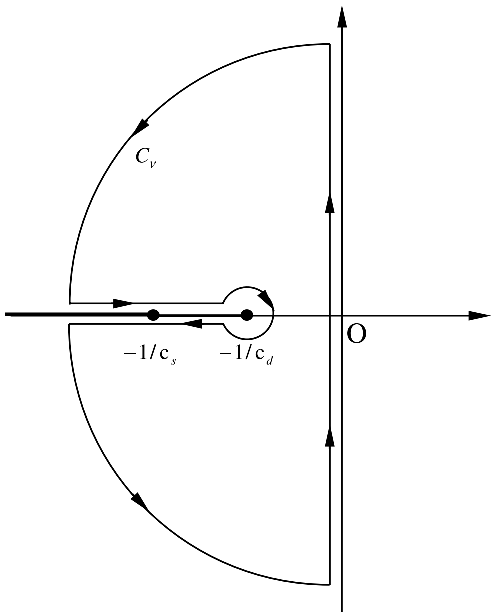

Fig. 5. Contour of integration in the complex p-plane to reduce the inner integral on the left-hand side of Eq. (29). The rays of the two quarter-circles tend at infinity. The case $\omega - x > 0$ is represented.

The right-hand side of Eq. (E.5) is nothing else but the one-sided Laplace transform over time of

$$\frac{1}{\pi}\frac{1}{v-x}\mathrm{Im}[\Sigma_+(p(t))]H(t-\eta(v)). \tag{E.7}$$

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